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Some conformally-flat space–times of a divergence-free Riemann tensor

Larry J Rose, Allan E G Stuart and Gordon W Barrett

Department of Mathematics, The City University, St John Street, London EC1V 4PB, UK

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Abstract. We present some interesting, conformally-flat metrics for a pseudo-Riemannian space–time with a divergence-free curvature tensor. The effective (gravitational) potentials for these metrics show both repulsive and periodic properties.

From an experimental point of view Einstein's theory of gravity is the most successful model of gravitational phenomena that we have at the present time. However, its theoretical structure, though both simple and elegant, has two major disadvantages. In the first place, it is an isolated description. All attempts at unifying the principles of general relativity with those of classical electromagnetism and quantum theory have resulted in formulations which are either ambiguous, highly contrived or both. Secondly, it is an essentially singular description and leads inevitably to gravitational collapse with all its attendant pathologies. For example, the bizarre extrapolations of the consequences of the singularity theorems are in sharp contrast to the 'physically reasonable' assumptions needed to prove them (for details see Hawking 1976, Carter 1968).

These undesirable properties of the Einstein model have led to a search for viable alternatives which are either non-singular, more amenable to quantisation or, hopefully, possess both these features. A recent attempt in this direction was made by Yang (1974) who proposed the free-field equations

$$R_{\mu\nu;\sigma} = R_{\mu\sigma;\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor and the semicolon denotes the covariant derivative. It should be noted, however, that this is not the first time these equations have appeared. They had previously occurred in a theory put forward by Stephenson (1958) and were first studied, in the context of a Riemannian connection, by Kilmister and Newman (1961), and Kilmister (1962). On the other hand, Yang's derivation is novel, since he treats gravity as a conventional non-Abelian gauge field and recent developments in quantum field theory have shown that considerable progress has been made in devising renormalisation procedures for non-Abelian gauge fields. Thus, it is conceivable that this theory can be quantised and may perhaps lead to more tractable results than the Einstein model.

At the formal level, the Yang equations are a generalisation, to third differential order, of Einstein's free-field equations $R_{\mu\nu} = 0$ and hence, they represent a model with extra degrees of freedom: degrees of freedom which are frozen in the Einstein

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case. As a direct result, the model has a large variety of solution space–times, including, of course, the Einstein spaces, and the purpose of this paper is to present some interesting, conformally-flat solutions of (1) for the case of a static, isotropic metric. Note, in this context, that the Einstein equations have no conformally-flat solution space–times other than the trivial one of the Minkowski metric.

We should mention, at this point, that there are certain misgivings about the consistency of Yang's derivation (see Fairchild 1976, especially reference 12 at the end of his paper). However, these seem to centre around the complete—or, rather, incomplete—application of a variational principle, the use of which, in this particular context, is generally considered to be dubious anyway (see Stephenson 1977). Thus, Fairchild's arguments are also open to question. We leave the reader to make up his own mind on this matter, but in our opinion the debate is somewhat premature. A variational principle, often constructed '*a posteriori*', is one of a number of heuristic devices employed for selecting mathematical models and should not be used, in the first instance, to validate or invalidate any particular model. This function is accomplished by developing the consequences of the model and hence providing it with a predictive data base. It is towards this latter end that our paper is directed. Nevertheless, for those who like some sort of non-controversial justification for considering the model, note that equations (1) are equivalent to demanding a pseudo-Riemannian space–time whose curvature tensor is everywhere divergence free. This follows directly from the contracted Bianchi identities and is, effectively, the necessary and sufficient condition that Yang uses to define his 'pure spaces'. One consequence of this is that the scalar curvatures of all solutions of (1) are necessarily constants. Thus, in this model, free-field gravity is a manifestation of a conserved curvature tensor rather than the stronger Einstein condition of a zero Ricci tensor. In this sense the model is also mathematically interesting.

We start with (1) and consider a static, isotropic metric in the 'standard' space-like form

$$ds^2 = f(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - g(r)\,dt^2, \tag{2}$$

where the functions $f(r)$ and $g(r)$ are determined by the requirement that (2) be a solution of (1). In a previous publication (Barrett *et al* 1977) we have shown that (2) will be a solution of (1) provided that f and g satisfy the minimal differential equations

$$fg' - gf' = \frac{f^2 g}{3}\left(rR + \frac{3C}{r^2}\right), \tag{3a}$$

$$\left(\frac{f'}{f}\right)' + \frac{1}{6}\left(rR + \frac{3C}{r^2}\right)\left[3f' + \left(\frac{rR}{3} + \frac{C}{r^2}\right)f^2\right] + \frac{2}{r^2}(1-f) = 0, \tag{3b}$$

where R is the (constant) curvature scalar and C is, what we have termed, the conformal constant. Putting $R = 0$ and $C = 4m$ in (3) reduces them to the minimal differential equations of the Einstein model for which, of course, the Schwarzschild space–time is the unique solution.

For conformally-flat solutions of (3) C has to be zero (Barrett *et al* 1977), and we initially consider those particular space–times for which $R = 0$ as well. In this case (3a) gives us $g = \alpha f$, where α is a constant, while (3b) reduces to

$$r^2 ff'' - (rf')^2 + 2f^2(1-f) = 0. \tag{4}$$

Now (4) is an equation of polynomial class (Davis 1962) and to solve it we look for a linear fractional transformation of the dependent variable f which will reduce it to one of the 50 soluble types listed by Davis. The general transformation takes the form

$$f=\frac{a+bz}{c+dz}, \tag{5}$$

where a, b, c and d are constants or free functions of r and z is the new dependent variable. After much tedious algebra we find that the choice $a=0$, $b=r^2$, $c=1$, $d=0$ gives us the desired result and reduces (4) to

$$zz''=(z')^2+2z^3. \tag{6}$$

Now, by putting $z=\mathrm{e}^u$, we get

$$u''=2\mathrm{e}^u, \tag{7}$$

which can be integrated once to give

$$(u')^2=4\mathrm{e}^u+2k_1, \tag{8}$$

where k_1 is the constant of integration. The further substitution $2p^2=2\mathrm{e}^u+k_1$ then leads to the explicit integral

$$\int\frac{2\,\mathrm{d}p}{2p^2-k_1}=\pm(r+k_2) \tag{9}$$

where k_2 is a second constant of integration. We now consider three cases.

(*i*) $k_1=0$

Equation (9) can be integrated to give

$$g=\alpha f=\alpha[1+(k_2/r)]^{-2} \tag{10a}$$

which is a generalisation of a solution of (1) first given by Pavelle (1975) and by Thompson (1975). The metric given by (10*a*) is obviously asymptotically flat and, if we interpret the resulting space–time as that of an isolated point mass m, then Newtonian correspondence gives us $k_2=m$. However, this space–time has bad post-Newtonian behaviour. There is no asymptotic bending of light and perihelion motion is in the opposite direction to that observed. On the other hand, the effective potential of the space–time for a test particle with angular momentum L (in the usual units) is given by

$$V_{\mathrm{eff}}^2\sim(1+m/r)^{-2}(1+L^2/r^2), \tag{10b}$$

which shows redeeming features in the strong-field region. We have analysed this metric in detail (Stuart *et al* 1977, submitted for publication, preprints available on request) and shall report on it elsewhere.

(*ii*) $k_1<0$

In this case, on integrating equation (9), we find

$$2g=2\alpha f=\alpha|k_1|r^2\sec^2[(|k_1|/2)^{1/2}(r+k_2)]. \tag{11a}$$

The resulting metric appears to be highly pathological, since in addition to its periodic properties it also diverges when $r+k_2 \to (2n+1)\pi/2$. If we write down the effective potential of the corresponding space–time we find, for time-like geodesics,

$$V_{\text{eff}}^2 \sim (r^2 \sec^2 r)(1+L^2/r^2) \tag{11b}$$

where, without loss of generality, we have chosen $k_1 = -2$ and $k_2 = 0$. The space–time thus consists of a series of inescapable 'traps' which seem to have more in common with quantum mechanics than with gravitation.

(iii) $k_1 > 0$

Once more, equation (9) can be integrated to give

$$2g = 2\alpha f = \alpha k_1 r^2 \operatorname{cosech}^2[(k_1/2)^{1/2}(r+k_2)]. \tag{12a}$$

This metric is rather interesting. Firstly, notice that as $r \to \infty$ both f and $g \to 0$ irrespective of the value of k_2. Thus, in this limit, the space–time represented by (12*a*) degenerates into a (space-like) two-sphere. On the other hand, the behaviour of the metric near $r=0$ depends critically on the value of k_2. For example, if $k_2=0$ then the space–time becomes flat as $r \to 0$. We can get a better idea of these properties by looking at the effective potential for timelike geodesics

$$V_{\text{eff}}^2 \sim (r^2+L^2) \operatorname{cosech}^2(r+k_2), \tag{12b}$$

where, for convenience, we have put $k_1 = 2$. We can now examine the three cases separately.

(*a*) If $k_2 = 0$ then the potential is repulsive everywhere and only radial geodesics with energy $E \geq 1$ can get through to the origin, $r=0$, where the space–time is flat. The origin is therefore an unstable equilibrium point for radial test particles with $E=1$.

(*b*) If $k_2 < 0$ the potential becomes infinite for $r = |k_2|$, is repulsive for $r > |k_2|$ and attractive for $r < |k_2|$. The potential thus takes the form of a closed spherical 'box' of radius $|k_2|$ with a stable equilibrium point at $r=0$. As a result, anything inside cannot get out and vice versa.

(*c*) If $k_2 > 0$ then, for $L > 0$, the potential is similar to that of case (*a*). For $L=0$ the maximum of the potential moves to a non-zero value of r while the origin becomes a minimum.

Although this metric is obviously not applicable in the weak-field region, it could be useful in those regions where the Einstein model becomes singular. One can envisage a 'compromise' between the Schwarzschild metric and this metric which is non-singular and still satisfies the weak-field tests.

The metrics discussed above exhaust the set of conformally-flat space–times (of the form (2)) which are solutions of (1) and have zero scalar curvature. For $R \neq 0$ equation (3*b*) becomes

$$r^2 ff'' - (rf')^2 + (R/2) r^3 f^2 f' + 2f^2(1-f) + (R^2/18)(rf)^4 = 0, \tag{13}$$

which is, again, an equation of polynomial class. However, we have not succeeded in finding a linear fractional transformation which will reduce (13) to one of the 50 soluble types mentioned before, and are inclined to believe that, even though the equation is of polynomial class, it cannot be solved by this method.

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